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by

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Minimax Estimation of a Multivariate Normal Mean under a Convex Loss Function

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Summary

Let $\underset{\sim}{X} = (X_1, \ldots, X_p)' \sim N_p(\underset{\sim}{\mu}, \Sigma)$ where $\underset{\sim}{\mu} = (\mu_1, \ldots, \mu_p)'$ and $\Sigma = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_p^2)$ are both unknown and $p \geq 3$. Let $(n_i-2)w_i/\sigma_i^2 \sim \chi^2_{n_i}$, independent of w_j $(i \neq j = 1, \ldots, p)$. Assume that $(w_1, \ldots, w_p)$ and $\underset{\sim}{X}$ are independent. Define $W = \mathrm{diag}(w_1, \ldots, w_p)$ and $\|\underset{\sim}{X}\|_W^2 = \underset{\sim}{X}'W^{-1}Q^{-1}W^{-1}\underset{\sim}{X}$ where $Q = \mathrm{diag}(q_1, \ldots, q_p), q_i > 0, i = 1, \ldots, p$. In this paper, the minimax estimator of Berger and Bock (Ann. Statist. 4 (1976), 642-648), given by $\underset{\sim}{\delta}(\underset{\sim}{X}, W) = [I_p - r(\underset{\sim}{X}, W)\|\underset{\sim}{X}\|_W^{-2}Q^{-1}W^{-1}]\underset{\sim}{X}$, is shown to be minimax relative to the convex loss $(\underset{\sim}{\delta}-\underset{\sim}{\mu})'[\alpha Q + (1-\alpha)\Sigma^{-1}](\underset{\sim}{\delta}-\underset{\sim}{\mu})/C$, where $C = \alpha\, \mathrm{tr}(\Sigma Q) + (1-\alpha)p$ and $0 \leq \alpha \leq 1$, under certain conditions on $r(\underset{\sim}{X}, W)$. This generalizes the above-mentioned result of Berger and Bock.

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1. <u>Introduction</u>. Let $\underset{\sim}{X} = (X_1, \ldots, X_p)'$ be a p-variate ($p \geq 3$) random vector normally distributed with mean $\underset{\sim}{\mu} = (\mu_1, \ldots, \mu_p)'$ and covariance matrix $\Sigma = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_p^2)$ where σ_i^2, $i = 1, \ldots, p$, are unknown. This paper obtains a class of minimax estimators for $\underset{\sim}{\mu}$ when the loss incurred in estimating $\underset{\sim}{\mu}$ by $\underset{\sim}{\delta}$ is given by

$$(1.1) \qquad L(\underset{\sim}{\delta}; \underset{\sim}{\mu}, \Sigma) = (\underset{\sim}{\delta}-\underset{\sim}{\mu})'[\alpha Q + (1-\alpha)\Sigma^{-1}](\underset{\sim}{\delta}-\underset{\sim}{\mu})/C,$$

where $0 \leq \alpha \leq 1$, $C = \alpha\, \mathrm{tr}(Q\Sigma) + (1-\alpha)p$, and Q is a known $p\times p$ diagonal matrix with diagonal elements $q_i > 0$, $i = 1, \ldots, p$. The loss (1.1) is a convex combination of two commonly used loss functions, namely

$$(1.2) \qquad L_1(\underset{\sim}{\delta}; \underset{\sim}{\mu}, \Sigma) = (\underset{\sim}{\delta}-\underset{\sim}{\mu})'Q(\underset{\sim}{\delta}-\underset{\sim}{\mu})/\mathrm{tr}(Q\Sigma)$$

and

$$(1.3) \qquad L_2(\underset{\sim}{\delta}; \underset{\sim}{\mu}, \Sigma) = (\underset{\sim}{\delta}-\underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{\delta}-\underset{\sim}{\mu})/p.$$

The loss function $L_1(\underset{\sim}{\delta}; \underset{\sim}{\mu}, \Sigma)$ may be used when the relative importance of the parameters to be estimated is reflected by a known set of weights represented by Q, while the loss $L_2(\underset{\sim}{\delta}; \underset{\sim}{\mu}, \Sigma)$ represents the case when the relative importance is reflected naturally by the inverse of the covariance matrix of the variables. In the literature, statisticians tend to use either $L_1(\underset{\sim}{\delta}; \underset{\sim}{\mu}, \Sigma)$ or $L_2(\underset{\sim}{\delta}; \underset{\sim}{\mu}, \Sigma)$ as a loss function. In the convex loss (1.1) we have combined both viewpoints in such a way that the more α is near 1 the more loss is to be assessed by $L_1(\underset{\sim}{\delta}; \underset{\sim}{\mu}, \Sigma)$, and vice versa.

It is clear that the maximum likelihood estimator, $\underset{\sim}{X}$, is minimax with risk $R(\underset{\sim}{X}, \underset{\sim}{\mu}) = E_{\underset{\sim}{X}} L(\underset{\sim}{X}; \underset{\sim}{\mu}, \Sigma) = 1$. Thus an estimator $\underset{\sim}{\delta}$ will be minimax under loss (1.1) if and only if $R(\underset{\sim}{X}, \underset{\sim}{\mu}) - R(\underset{\sim}{\delta}, \underset{\sim}{\mu}) \geq 0$ for all $\underset{\sim}{\mu}$ and Σ. In evaluating the difference in risk it is necessary to impose certain conditions on Σ. Specifically, the values or bounds of the trace of $\Sigma^{-1}Q^{-1}$, denoted by $\mathrm{tr}(\Sigma^{-1}Q^{-1})$, and the minimum characteristic root of $Q\Sigma$, denoted by $\mathrm{ch}_{\min}(Q\Sigma)$, will be assumed. Similar conditions have been noted by various authors. For example, Gleser (1976) shows that if a lower bound for $\mathrm{ch}_{\min}(Q\Sigma)$ is known then a family of minimax estimators for $\underset{\sim}{\mu}$ can be obtained under the loss function $L_1(\underset{\sim}{\delta}; \underset{\sim}{\mu}, \Sigma)$; otherwise no estimator of the form $[I_p - h(\underset{\sim}{X}'W^{-1}\underset{\sim}{X})Q^{-1}W^{-1}]\underset{\sim}{X}$ can be minimax for $\underset{\sim}{\mu}$ unless $h(u) = 0$ for almost all $u \geq 0$ where $W \sim W(n, \Sigma)$ independent of $\underset{\sim}{X}$ and $h: R \to R$ is a real function satisfying certain conditions. In our case where both Q and Σ are diagonal matrices a sufficient set of conditions on Σ would be

$$(1.4) \qquad \max_{1\leq i\leq p} \{\sigma_i^2\} \leq 1/c_1 \quad \text{and} \quad \min_{1\leq i\leq p} \{\sigma_i^2\} \geq 1/c_2$$

for some positive constants c_1 and c_2 $(0 < c_1 \leq c_2 < \infty)$. This set of conditions on Σ is not unreasonable in application.

2. Useful Lemmas. In establishing the minimaxity of an estimator of the mean of a multivariate normal distribution the following lemmas are very useful. They are presented here for ease of reference and without proof.

Lemma 2.1. [Stein (1974)]. Let $Y \sim N(0, 1)$ and let g be an absolutely continuous function, $g: R \to R$. Then

$$E_Y[g'(Y)] = E_Y[Yg(Y)]$$

provided that all expectations exist and are finite.

Lemma 2.2. [Efron and Morris (1976)]. Let $U \sim \chi_n^2$ and let g be as defined in Lemma 2.1. Then

$$E_U[Ug(U)] = nE_U[g(U)] + 2E_U[Ug'(U)]$$

provided that all expectations exist and are finite.

Corollary 2.2.1. Let U and g be as defined in Lemma 2.2. Let $Z = cU/(n-2)$, $c > 0$, and $h(Z) = g[(n-2)Z/c]$. Then

$$E_Z[(n-2)Zh(Z)/c] = nE_Zh(Z) + 2E_Z[Zh'(Z)]$$

provided that all expectations exist and are finite.

Lemma 2.3. [Lehmann (1966)]. Let S be any random variable, and let $p_1(S)$ and $p_2(S)$ map the real line into itself. If $p_1(S)$ and $p_2(S)$ are either both nonincreasing in S or both nondecreasing in S, then

$$E_S[p_1(S)p_2(S)] \geq E_S[p_1(S)]E_S[p_2(S)].$$

Note that Lemmas 2.1 and 2.2 may be proved by integration by parts and the corollary by the indicated change of variable. Lemma 2.3 follows from a new concept of dependency, namely the positive quadrant dependence, introduced by Lehmann (1966). The above results will be employed in the proof of the main theorem in the next section.

3. Main Result. Let $\underset{\sim}{X}$ be a p-variate ($p \ge 3$) random vector normally distributed with unknown mean $\underset{\sim}{\mu}$ and unknown covariance matrix $\Sigma = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_p^2)$. Assume that $(n_i-2)w_i/\sigma_i^2 \sim \chi^2_{n_i}$, $(n_i > 2)$, $i = 1, \ldots, p$, where w_i and w_j ($i \ne j = 1, \ldots, p$) are mutually independent and are independent of $\underset{\sim}{X}$. As in Berger and Bock (1976), define $W = \mathrm{diag}(w_1, \ldots, w_p)$, $T = \min_{1\le i\le p} (\chi^2_{n_i}/n_i)$, and $\tau = \tau(n_1, \ldots, n_p) = E(T^{-1})$. In the following theorem we will obtain a class of minimax estimators for $\underset{\sim}{\mu}$ relative to the loss function given by (1.1).

Theorem. The estimator

$$(3.1) \qquad \underset{\sim}{\delta}(\underset{\sim}{X}, W) = [I_p - r(\underset{\sim}{X}, W)\|\underset{\sim}{X}\|_W^{-2}Q^{-1}W^{-1}]\underset{\sim}{X}$$

is minimax for $\underset{\sim}{\mu}$ relative to the loss function (1.1) with $Q = \mathrm{diag}(q_1, \ldots, q_p)$, where $\|\underset{\sim}{X}\|_W^2 = \underset{\sim}{X}'W^{-1}Q^{-1}W^{-1}\underset{\sim}{X}$, provided that the following conditions are satisfied

(i) $0 \le r(\underset{\sim}{X}, W) \le \dfrac{2\{\alpha(p-2\tau) + (1-\alpha)[\mathrm{tr}(\Sigma^{-1}Q^{-1}) - 2\tau \mathrm{ch}_{max}(\Sigma^{-1}Q^{-1})]\}}{\alpha + (1-\alpha)\mathrm{ch}_{max}(\Sigma^{-1}Q^{-1})}$ with $\alpha(p-2\tau) + (1-\alpha)[\mathrm{tr}(\Sigma^{-1}Q^{-1}) - 2\tau \mathrm{ch}_{max}(\Sigma^{-1}Q^{-1})] \ge 0$ and $0 \le \alpha \le 1$,

(ii) $r(\underset{\sim}{X}, W)$ is nondecreasing in $|X_i|$, $i = 1, \ldots, p$,

(iii) $r(\underset{\sim}{X}, W)$ is nonincreasing in w_i, $i = 1, \ldots, p$,

(iv) $r(\underset{\sim}{X}, W)\|\underset{\sim}{X}\|_W^{-2}$ is nondecreasing in w_i, $i = 1, \ldots, p$.

Proof. Write $\underset{\sim}{\delta} = \underset{\sim}{\delta}(\underset{\sim}{X}, W)$ and $r = r(\underset{\sim}{X}, W)$. Let $\Delta = R(\underset{\sim}{X}, \underset{\sim}{\mu}) - R(\underset{\sim}{\delta}, \underset{\sim}{\mu})$. Then

$$(3.2) \qquad C\Delta = E_{W,\underset{\sim}{X}}\{2r\|\underset{\sim}{X}\|_W^{-2}(\underset{\sim}{X}-\underset{\sim}{\mu})'[\alpha Q + (1-\alpha)\Sigma^{-1}]Q^{-1}W^{-1}\underset{\sim}{X}$$

$$-r^2\|\underset{\sim}{X}\|_W^{-4}\underset{\sim}{X}'W^{-1}Q^{-1}[\alpha Q + (1-\alpha)\Sigma^{-1}]Q^{-1}W^{-1}\underset{\sim}{X}\}$$

$$= 2[\alpha\Delta_1 + (1-\alpha)\Delta_2], \text{ say,}$$

where

$$\Delta_1 = E_{W,\underset{\sim}{X}}[r\|\underset{\sim}{X}\|_W^{-2}(\underset{\sim}{X}-\underset{\sim}{\mu})'W^{-1}\underset{\sim}{X} - r^2\|\underset{\sim}{X}\|_W^{-2}/2]$$

and

$$\Delta_2 = E_{W,\underset{\sim}{X}}[r\|\underset{\sim}{X}\|_W^{-2}(\underset{\sim}{X}-\underset{\sim}{\mu})'\Sigma^{-1}Q^{-1}W^{-1}\underset{\sim}{X} - r^2\|\underset{\sim}{X}\|_W^{-4}\underset{\sim}{X}'W^{-1}Q^{-1}\Sigma^{-1}Q^{-1}W^{-1}\underset{\sim}{X}].$$

In the following we will find appropriate lower bounds for Δ_1 and Δ_2 which, in conjunction with Condition (i), will establish that $\Delta \geq 0$. But a lower bound for Δ_1 has been obtained by Berger and Bock (1976, Eq. (2.10)), namely

$$\Delta_1 \geq E_{\underset{\sim}{X}}\{E_W(r\|\underset{\sim}{X}\|_W^{-2})[p - 2\tau - E_W(r/2)]\}. \tag{3.3}$$

We will proceed to find a lower bound for Δ_2. Let

$$\Delta_2 = \Delta_{21} + \Delta_{22} \tag{3.4}$$

where

$$\Delta_{21} = E_{W,\underset{\sim}{X}}[r\|\underset{\sim}{X}\|_W^{-2}(\underset{\sim}{X}-\underset{\sim}{\mu})'\Sigma^{-1}Q^{-1}W^{-1}\underset{\sim}{X}] \tag{3.5}$$

and

$$\begin{aligned} \Delta_{22} &= E_{W,\underset{\sim}{X}}[-(1/2)r^2\|\underset{\sim}{X}\|_W^{-4}\underset{\sim}{X}'W^{-1}Q^{-1}\Sigma^{-1}Q^{-1}W^{-1}\underset{\sim}{X}] \\ &\geq E_{W,\underset{\sim}{X}}[-(1/2)r^2\|\underset{\sim}{X}\|_W^{-2}ch_{max}(\Sigma^{-1}Q^{-1})]. \end{aligned} \tag{3.6}$$

Then, taking expectation with respect to $\underset{\sim}{X}$ first and then W, and by an application of Lemma 2.1, we have

$$(3.7)\qquad \Delta_{21} = E_{W,\underset{\sim}{X}}\left[\sum_{i=1}^{p} \frac{1}{\sigma_i q_i w_i} \left(r\|\underset{\sim}{X}\|_W^{-2} X_i\right)\left(\frac{X_i-\mu_i}{\sigma_i}\right)\right]$$

$$= E_{W,\underset{\sim}{X}}\left[\sum_{i=1}^{p} \frac{1}{q_i w_i} \frac{\partial}{\partial X_i} \left(r\|\underset{\sim}{X}\|_W^{-2} X_i\right)\right]$$

$$= E_{W,\underset{\sim}{X}}\left[\sum_{i=1}^{p}\left(\frac{r}{\|\underset{\sim}{X}\|_W^2} - \frac{2rX_i^2}{\|\underset{\sim}{X}\|_W^4 q_i w_i^2} + \frac{X_i}{\|\underset{\sim}{X}\|_W^2} \frac{\partial r}{\partial X_i}\right)\right]$$

$$\geq E_{W,\underset{\sim}{X}}\left[\frac{r}{\|\underset{\sim}{X}\|_W^2} \sum_{i=1}^{p} \frac{1}{q_i w_i} - \frac{2r}{\|\underset{\sim}{X}\|_W^4} \sum_{i=1}^{p} \frac{X_i^2}{q_i^2 w_i^3}\right].$$

The last inequality follows from Condition (ii) since $X_i(\partial r/\partial X_i) \geq 0$ for all $i = 1, \ldots, p$. The first term in the right-hand side of (3.7) may be further evaluated by taking the expectation first with respect to W and then with respect to $\underset{\sim}{X}$. Recall that $(n_i-2)w_i/\sigma_i^2 \sim \chi^2_{n_i}$. Then, it follows from Corollary 2.2.1 with $h(w_i) = r/(\|\underset{\sim}{X}\|_W^2 w_i)$ that, for each $i = 1, \ldots, p$,

$$n_i E_{w_i}\left(\frac{r}{\|\underset{\sim}{X}\|_W^2 w_i}\right) = \frac{n_i-2}{\sigma_i^2} E_{w_i}\left(\frac{r}{\|\underset{\sim}{X}\|_W^2}\right) - 2E_{w_i}\left[\frac{1}{\|\underset{\sim}{X}\|_W^2} \frac{\partial r}{\partial w_i} + \frac{2rX_i^2}{\|\underset{\sim}{X}\|_W^4 q_i w_i^3} - \frac{r}{\|\underset{\sim}{X}\|_W^2 w_i}\right].$$

Therefore,

$$(3.8)\quad E_{W,\underset{\sim}{X}}\left(\frac{r}{\|\underset{\sim}{X}\|_W^2 w_i}\right) = E_{W,\underset{\sim}{X}}\left[\frac{r}{\|\underset{\sim}{X}\|_W^2 \sigma_i^2} - \frac{4rX_i^2}{(n_i-2)\|\underset{\sim}{X}\|_W^4 q_i w_i^3} - \frac{2}{(n_i-2)\|\underset{\sim}{X}\|_W^2} \frac{\partial r}{\partial w_i}\right]$$

$$\geq E_{W,\underset{\sim}{X}}\left[\frac{r}{\|\underset{\sim}{X}\|_W^2 \sigma_i^2} - \frac{4rX_i^2}{(n_i-2)\|\underset{\sim}{X}\|_W^4 q_i w_i^3}\right].$$

The last inequality follows by Condition (iii). Now, substituting (3.8) into the last expression of (3.7), we have

$$(3.9)\quad \Delta_{21} \ge E_{W,\underset{\sim}{X}}\left[\frac{r}{\|\underset{\sim}{X}\|_W^2}\sum_{i=1}^{p}\frac{1}{\sigma_i^2 q_i} - \frac{4r}{\|\underset{\sim}{X}\|_W^4}\sum_{i=1}^{p}\frac{X_i^2}{(n_i-2)q_i^2 w_i^3} - \frac{2r}{\|\underset{\sim}{X}\|_W^4}\sum_{i=1}^{p}\frac{X_i^2}{q_i^2 w_i^3}\right]$$

$$= E_{W,\underset{\sim}{X}}\left[\frac{r}{\|\underset{\sim}{X}\|_W^2}\sum_{i=1}^{p}\frac{1}{\sigma_i^2 q_i} - \frac{2r}{\|\underset{\sim}{X}\|_W^4}\sum_{i=1}^{p}\left(\frac{n_i}{n_i-2}\right)\frac{X_i^2}{q_i^2 w_i^3}\right]$$

$$\ge E_{W,\underset{\sim}{X}}\left[\frac{r}{\|\underset{\sim}{X}\|_W^2}\sum_{i=1}^{p}\frac{1}{\sigma_i^2 q_i} - \frac{2r}{\|\underset{\sim}{X}\|_W^2}\max_{1\le i\le p}\left(\frac{n_i}{n_i-2}\,\frac{\sigma_i^2}{w_i}\,\frac{1}{\sigma_i^2 q_i}\right)\right]$$

$$\ge E_{W,\underset{\sim}{X}}\{(r\|\underset{\sim}{X}\|_W^{-2})[\mathrm{tr}(\Sigma^{-1}Q^{-1}) - 2T^{-1}\mathrm{ch}_{\max}(\Sigma^{-1}Q^{-1})]\}.$$

Substituting (3.9) and (3.6) into (3.4), we obtain

$$(3.10)\quad \Delta_2 \ge E_{W,\underset{\sim}{X}}\{(r\|\underset{\sim}{X}\|_W^{-2})[\mathrm{tr}(\Sigma^{-1}Q^{-1})-2T^{-1}\mathrm{ch}_{\max}(\Sigma^{-1}Q^{-1})-(r/2)\mathrm{ch}_{\max}(\Sigma^{-1}Q^{-1})]\}$$

$$\ge E_{\underset{\sim}{X}}\{E_W(r\|\underset{\sim}{X}\|_W^{-2})[\mathrm{tr}(\Sigma^{-1}Q^{-1})-2\tau\mathrm{ch}_{\max}(\Sigma^{-1}Q^{-1})-E_W(r/2)\mathrm{ch}_{\max}(\Sigma^{-1}Q^{-1})]\}.$$

The last inequality in (3.10) is obtained by an application of Lemma 2.3 since $r\|\underset{\sim}{X}\|_W^{-2}$ is nondecreasing in w_i and $[\mathrm{tr}(\Sigma^{-1}Q^{-1}) - 2T^{-1}\mathrm{ch}_{\max}(\Sigma^{-1}Q^{-1}) - (r/2)\mathrm{ch}_{\max}(\Sigma^{-1}Q^{-1})]$ is also nondecreasing in w_i for $i = 1, \ldots, p$, and since w_i and w_j $(i \ne j)$ are mutually independent. Hence, upon the substitution of (3.10) and (3.3) into (3.2), we conclude that

$$(3.11)\quad C\Delta/2 = \alpha\Delta_1 + (1-\alpha)\Delta_2$$

$$\ge E_{\underset{\sim}{X}}\{E_W(r\|\underset{\sim}{X}\|_W^{-2})$$

$$\times \{\alpha(p-2\tau) + (1-\alpha)[\mathrm{tr}(\Sigma^{-1}Q^{-1}) - 2\tau\ \mathrm{ch}_{\max}(\Sigma^{-1}Q^{-1})]$$

$$- [\alpha + (1-\alpha)\mathrm{ch}_{\max}(\Sigma^{-1}Q^{-1})]E_W(r/2)\}\}.$$

By Condition (i), it is clear that the last expression of (3.11) is non-negative for all $\underset{\sim}{\mu}$ and Σ and hence $\underset{\sim}{\delta}$ is minimax proving the theorem.

Remarks. (1) Unless Condition (i) is satisfied, the theorem is vacuous. Thus it is required not only that $\mathrm{tr}(\Sigma^{-1}Q^{-1})$ and $\mathrm{ch}_{max}(\Sigma^{-1}Q^{-1})$ be known but that, for $0 \le \alpha \le 1$,

$$\alpha(p-2\tau) + (1-\alpha)[\mathrm{tr}(\Sigma^{-1}Q^{-1}) - 2\tau\ \mathrm{ch}_{max}(\Sigma^{-1}Q^{-1})] \ge 0. \tag{3.12}$$

On the other hand, if there exists positive constants c_1 and c_2 $(0 < c_1 \le c_2 < \infty)$ such that (1.4) holds or, equivalently,

$$\mathrm{ch}_{min}(\Sigma^{-1}) \ge c_1, \quad \mathrm{ch}_{max}(\Sigma^{-1}) \le c_2$$

and

$$p \ge 2\tau[\alpha + (1-\alpha)c_2\ \mathrm{ch}_{max}(Q^{-1})]/[\alpha + (1-\alpha)c_1\ \mathrm{ch}_{min}(Q^{-1})],$$

then the theorem remains true with Condition (i) replaced by

$$(\mathrm{i}') \quad 0 \le r(\underset{\sim}{X}, W) \le \frac{2\{\alpha(p-2\tau) + (1-\alpha)[c_1\mathrm{tr}(Q^{-1}) - 2\tau\ c_2\ \mathrm{ch}_{max}(Q^{-1})]\}}{\alpha + (1-\alpha)c_2\ \mathrm{ch}_{max}(Q^{-1})}$$

(2) Berger and Bock (1976) show that the estimator $\underset{\sim}{\delta}(\underset{\sim}{X}, W)$ given by (3.1) is minimax for $\underset{\sim}{\mu}$ under the loss $L_1(\underset{\sim}{\delta}; \underset{\sim}{\mu}, \Sigma)$ and that there exists a large positive integer N such that, for all $n_i \ge N$, $p-2\tau > 0$ and $p \ge 3$. Furthermore, they provide a formula for calculating τ when the n_i's are even. In this paper, we employ their method of proof to establish our result. It should be noted that our result reduces to theirs when $\alpha = 1$.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $\underset{\sim}{X} = (X_1, \ldots, X_p)' \sim N_p(\underset{\sim}{\mu}, \Sigma)$ where $\underset{\sim}{\mu} = (\mu_1, \ldots, \mu_p)'$ and $\Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma$ are both unknown and $p \geq 3$. Let $(n_i-2)w_i/\sigma_i^2 \sim \chi^2_{n_i}$, independent of w_j $(i \neq j = 1, \ldots,$ Assume that $(w_1, \ldots, w_p)$ and $\underset{\sim}{X}$ are independent. Define $W = \text{diag}(w_1, \ldots, w_p)$ and $\|\underset{\sim}{X}\|^2_W = \underset{\sim}{X}'W^{-1}Q^{-1}W^{-1}\underset{\sim}{X}$ where $Q = \text{diag}(q_1, \ldots, q_p)$, $q_i > 0$, $i = 1, \ldots, p$. In this paper, the minimax estimator of Berger and Bock (Ann. Statist. 4 (1976), 642-648), given by $\underset{\sim}{\delta}(\underset{\sim}{X}, W) = [I_p - r(\underset{\sim}{X}, W)\|\underset{\sim}{X}\|_W^{-2}Q^{-1}W^{-1}]\underset{\sim}{X}$, is shown to be minimax relative to the convex los $(\underset{\sim}{\delta}-\underset{\sim}{\mu})'[\alpha Q + (1-\alpha)\Sigma^{-1}](\underset{\sim}{\delta}-\underset{\sim}{\mu})/C$, where $C = \alpha\, \text{tr}(\Sigma Q) + (1-\alpha)p$ and $0 \leq \alpha \leq 1$, under certain conditions on $r(\underset{\sim}{X}, W)$. This generalizes the above-mentioned result of Berger and Bock.